# On the homotopy elements $h_0h_n$

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June 6, 2018

# Contents

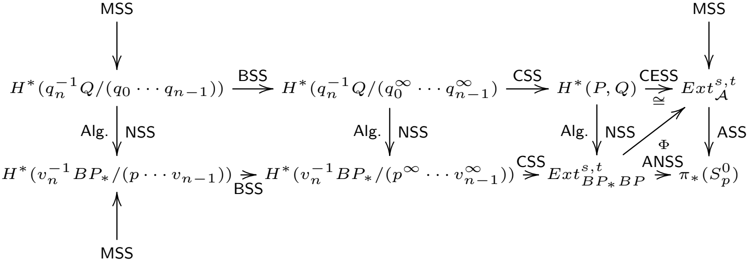
# Classical ASS and ANSS

- Let $p \geqslant 5$ be an odd prime. One has the classical Adams spectral sequence (ASS) and the Adams-Novikov spectral sequence (ANSS), they all converge to the stable homotopy groups of spheres.

$$\begin{array}{ccc} \{E_r^{s,t}, d_r\} \Longrightarrow \pi_*(S_p^0) & \qquad & E_2 = Ext_{BP_*BP}^{s,t}(BP_*, BP_*) \\ \downarrow \Phi & & \downarrow \Phi \\ \{E_r^{s,t}, d_r\} \Longrightarrow \pi_*(S_p^0) & & E_2 = Ext_{\mathcal{A}}^{s,t}(\mathbb{Z}/p, \mathbb{Z}/p) \end{array}$$

Between the ANSS and the ASS there is the Thom map $\Phi$ induced by $\Phi : BP \longrightarrow H\mathbb{Z}/p$.

To detect the $E_2$-terms of the ASS and of the ANSS, one has the following spectral sequences




where $P = \mathbb{Z}/p[\xi_1, \xi_2, \cdots]$ and $Q = \mathbb{Z}/p[q_0, q_1, \cdots]$.

# The homotopy elements $h_0h_n$

- One has $\beta_{p^n/p^n-1} \in Ext^{2,*}_{BP_*BP}(BP_*, BP_*)$, which is detected by the CSS and $\Phi(\beta_{p^n/p^n-1}) = h_0h_{n+1}$.

$$\begin{array}{ccccc} & & H^*(P,Q) & \xrightarrow[\cong]{\text{CESS}} & Ext^2_{\mathcal{A}} \\ & & \text{Alg.}\downarrow \text{NSS} & \nearrow \Phi & \downarrow \text{ASS} \\ H^0(v_2^{-1}BP_*/(p^\infty, v_1^\infty)) & \xrightarrow[\text{CSS}]{} & Ext^2_{BP_*BP} & \xrightarrow[\text{ANSS}]{} & \pi_*(S^0) \end{array}$$

$$\begin{array}{ccccc} \beta_{p^n/p^n-1} \in & Ext^{2,*}_{BP_*BP}(BP_*, BP_*) & \subset & \pi_*(BP \wedge \widetilde{X}_2) \\ \downarrow \Phi & \downarrow & & \downarrow \\ h_0h_{n+1} \in & Ext^{2,*}_{\mathcal{A}}(\mathbb{Z}/p, \mathbb{Z}/p) & \subset & \pi_*(H \wedge X_2) \end{array}$$

- The convergence of $h_0 h_{n+1}$ in the classical ASS (that of $\beta_{p^n/p^n-1}$ in the ANSS) have been being a long standing problem in stable homotopy groups of spheres.
- Let $M$ be the $mod\ p$ Moore spectrum, $M(1, p^n - 1)$ be the cofiber of $v_1^{p^n-1} : \Sigma^* M \longrightarrow M$.

**Secondary periodic family elements in the ANSS, D. Ravenel**

**Theorem** *Let $p \geqslant 5$ be an odd prime. If for some fixed $n \geqslant 1$,*

- *the spectrum $M(1, p^n - 1)$ is a ring spectrum,*
- *$\beta_{p^n/p^n-1}$ is a permanent cycle and*
- *the corresponding homotopy element has order $p$,*

*then $\beta_{sp^n/j}$ is a permanent cycle (and the corresponding homotopy element has order $p$) for all $s \geqslant 1$ and $1 \leqslant j \leqslant p^n - 1$.*

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- S. Oka proved that $M(1, p^n - 1)$ is a ring spectrum.
- From the theorem above and the convergence of $h_0 h_{n+1}$ one can prove the $\beta_{p^n/p^n-1}$ is a permanent cycle of order $p$.
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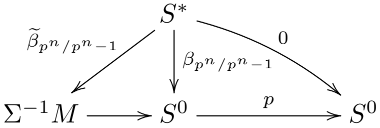


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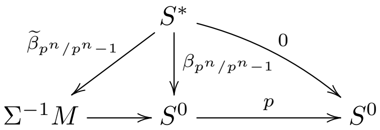
- 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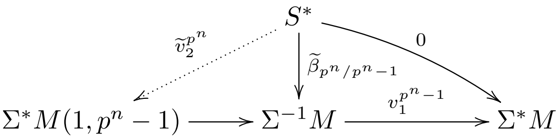

# Toda differential

- $\alpha_1$ and $b_0 = \beta_1$ in $Ext^{*,*}_{BP_*BP}(BP_*, BP_*)$ are permanent cycles in the ANSS, they converges to the homotopy elements $\alpha_1$, $\beta_1$ respectively.
- H. Toda proved that $\alpha_1\beta_1^p = 0$ in $\pi_*(S^0)$.
- The relation $\alpha_1\beta_1^p = 0$ support a Adams differential

$$d_r(x) = \alpha_1 b_0^p.$$

It is detected that $x = b_1$ i.e $d_{2p-1}(b_1)) = k \cdot \alpha_1 b_0^p$

- Based on $d_{2p-1}(b_1) = k \cdot \alpha_1 b_0^p$, D. Ravenel proved that

$$d_{2p-1}(b_n) \equiv \alpha_1 b_{n-1}^p$$

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- Consider the cofiber sequence

$$S^0 \xrightarrow{p} S^0 \longrightarrow M$$

which induces a short exact sequence of $BP$-homologies

$$0 \longrightarrow BP_* \xrightarrow{p} BP_* \longrightarrow BP_*M \longrightarrow 0$$

- The short exact sequence of $BP$-homologies induces a long exact sequence of $Ext$ groups and it commutes with the Adams differential:

$Ext^{s,t}_{BP_*BP}(BP_*, N)$ is denoted by $Ext^{s,t}(N)$ for short.

$$\begin{array}{ccccccccc}
\cdots \to & Ext^{1,*}(BP_*) & \to & Ext^{1,*}(BP_*M) & \xrightarrow{\delta} & Ext^{2,*}(BP_*) & \to \cdots \\
& \downarrow d_{2p-1} & & \downarrow d_{2p-1} & & \downarrow d_{2p-1} & \\
\cdots \to & Ext^{2p,*}(BP_*) & \to & Ext^{2p,*}(BP_*M) & \xrightarrow{\delta} & Ext^{2p+1,*}(BP_*) & \to \cdots
\end{array}$$

- There are elements $v_1 \in Ext^{0,*}(BP_*M)$, $h_{n+1} \in Ext^{1,*}(BP_*M)$, $v_1 b_{n-1}^p \in Ext^{2p,*}(BP_*M)$

$$\delta(h_{n+1}) = b_n, \qquad \delta(v_1 b_{n-1}^p) = \alpha_1 b_{n-1}^p$$
$$\delta(v_1 h_{n+1}) = \beta_{p^n/p^n-1}, \qquad \delta(v_1^2 b_{n-1}^p) = \alpha_2 b_{n-1}^p.$$

- So in the ANSS for the Moore spectrum one has

$$d_{2p-1}(h_{n+1}) = v_1 b_{n-1}^p, \qquad d_{2p-1}(v_1 h_{n+1}) = v_1^2 b_{n-1}^p.$$

- Applying the connecting homomorphism $\delta$, one has

$$d_{2p-1}(\beta_{p^n/p^n-1}) = \alpha_2 b_{n-1}^p.$$

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We could NOT prove that

$$\alpha_2 b_{n-1}^p \in Ext_{BP_*BP}^{2p+1,*}(BP_*, BP_*)$$

is non-zero in the $Ext$ groups although $\alpha_1 b_{n-1}^p$ is non-zero.

- $\alpha_2 b_0^p = 0$ because $\alpha_2\beta_1 = 0$. And we know that $\beta_{p/p-1}$ (resp. $h_0h_2$) survives to $E_\infty$

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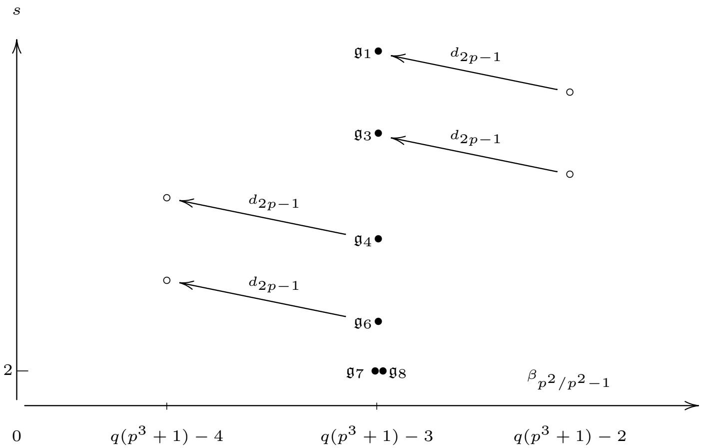
$s$
$\mathfrak{g}_1$
$d_{2p-1}$
$\mathfrak{g}_3$
$d_{2p-1}$
$d_{2p-1}$
$\mathfrak{g}_4$
$d_{2p-1}$
$\mathfrak{g}_6$
2
$\mathfrak{g}_7$ $\mathfrak{g}_8$
$\beta_{p^2/p^2-1}$
0
$q(p^3+1)-4$
$q(p^3+1)-3$
$q(p^3+1)-2$

# Small descent SS

- Let $T(m)$ be the Ravenel spectrum characterized by $BP_*T(m) = BP_*[t_1, t_2, \cdots, t_m]$. One has

$$S^0 \hookrightarrow T(1) \hookrightarrow T(2) \hookrightarrow \cdots \hookrightarrow T(m) \hookrightarrow \cdots \hookrightarrow BP$$

- Let $X$ be the $(p-1)q$ skeleton of $T(1)$, where $q = 2(p-1)$

$$X = S^0 \cup_{\alpha_1} e^q \cup_{\alpha_1} e^{2q} \cup \cdots \cup_{\alpha_1} e^{(p-1)q}$$

and let $\overline{X} = S^0 \cup_{\alpha_1} e^q \cup \cdots \cup_{\alpha_1} e^{(p-2)q}$ be the $(p-2)q$ skeleton of $T(1)$.

$$BP_*X = BP_*[t_1]/(t_1^p), \qquad BP_*\overline{X} = BP_*[t_1]/(t_1^{p-1})$$

- One has the cofiber sequences

$$
\begin{array}{ccccccc}
S^0 & \longrightarrow & X & \longrightarrow & \Sigma^q \bar{X} & \longrightarrow & S^{(p-1)q} \\
\Sigma^q \overline{X} & \longrightarrow & \Sigma^q X & \longrightarrow & S^{pq} & \longrightarrow & \Sigma^{pq} \overline{X} \\
S^{pq} & \longrightarrow & \Sigma^{pq} X & \longrightarrow & \Sigma^{(p+1)q} \overline{X} & \longrightarrow & S^{(2p-1)q} \\
 & & \cdots & & \cdots & &
\end{array}
$$

- The cofiber sequences gives raise short exact sequences of $BP_*$ homologies

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & BP_* & \longrightarrow & BP_* X & \longrightarrow & BP_* \Sigma^q \overline{X} & \longrightarrow & 0 \\
0 & \longrightarrow & BP\Sigma^q \overline{X} & \longrightarrow & BP_* \Sigma^q X & \longrightarrow & BP_* S^{pq} & \longrightarrow & 0 \\
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 & & \cdots & & & & \cdots & &
\end{array}$$

- From the short exact sequences, one gets a long exact sequence

$$0 \longrightarrow BP_* \longrightarrow BP_*X \longrightarrow BP_*\Sigma^q X \longrightarrow BP_*\Sigma^{pq} X \longrightarrow BP_*\Sigma^{(p+1)q} X \longrightarrow \cdots$$

and the long exact sequence induces the *small descent spectral sequence.*

**SDSS, D. Ravenel**

Let $X$ be as above. Then there is a spectral sequence converging to $Ext^{s+u,*}_{BP_*BP}(BP_*, BP_*)$ with $E_1$-term

$$E_1^{s,t,u} = Ext^{s,t}_{BP_*BP}(BP_*, BP_*X) \otimes E[\alpha_1] \otimes P[\beta_1]$$

where

$$E_1^{s,t,0} = Ext^{s,t}(BP_*X), \qquad \alpha_1 \in E_1^{0,q,1}, \qquad \beta_1 \in E_1^{0,pq,2}.$$

and $d_r : E_r^{s,t,u} \longrightarrow E_r^{s-r+1,t,u+r}$.

## D. Ravenel 1984

Let $p \geqslant 5$ be an odd prime, then with in $t - s < q(p^3 + p)$

$$Ext^{s,t}_{BP_*BP}(BP_*, BP_*X \otimes E_1^2) = A \oplus B \oplus C$$

where $\otimes E_1^2$ means except for the first periodic homotopy elements.

- Because the total degree $t-s$ of $\beta_1$ is $pq - 2 = 2p^2 - 2p - 2$ and that of $\beta_{p^2/p^2-1}$ is $4p - 2 \quad mod \quad pq - 2$

$$\begin{array}{r l}
 & p^2 \qquad\quad + 1 \\
2p^2 - 2p - 2 & \sqrt{2p^4 - 2p^3 \qquad + 2p - 4} \\
 & 2p^4 - 2p^3 - 2p^2 \\
\hline
 & 2p^2 + 2p - 4 \\
 & 2p^2 - 2p - 2 \\
\hline
 & 4p - 2
\end{array}$$

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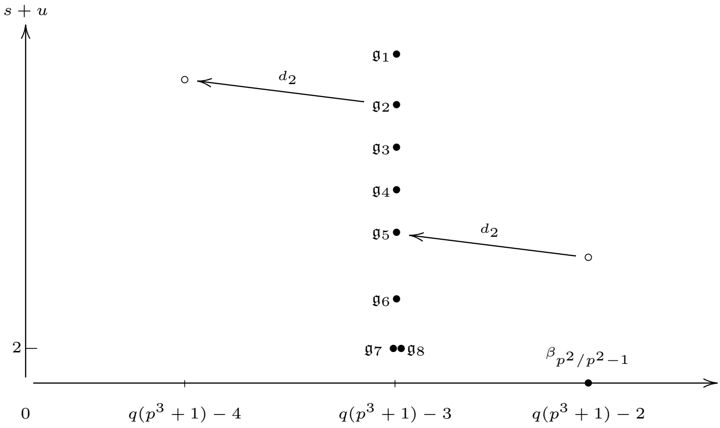
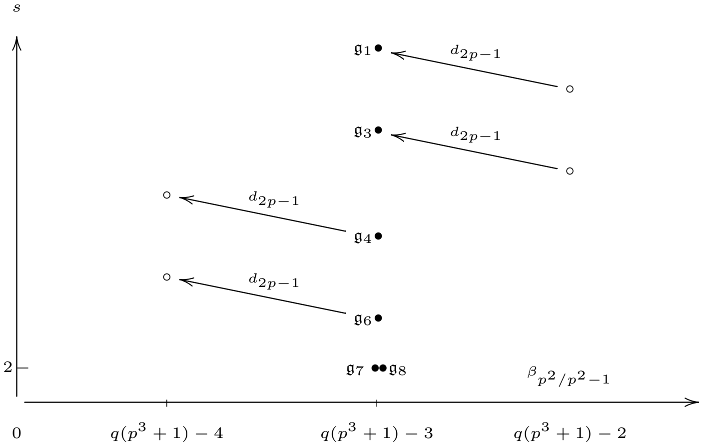
$$\begin{array}{r|l} & p^2 \qquad\quad +1 \\ 2p^2-2p-2 & 2p^4-2p^3 \qquad +2p-4 \\ & 2p^4-2p^3-2p^2 \\ \hline & 2p^2+2p-4 \\ & 2p^2-2p-2 \\ \hline & 4p-2 \end{array}$$

We computed the total degree of the generators in $(A \oplus B \oplus C) \otimes E[\alpha_1]$ $mod\ \ pq - 2$. From which we get the $E_1$-term of SDSS



Then we computed the Adams differential and get $d_r(\beta_{p^2/p^2-1}) = 0$.



# Further consideration, where is $\beta_{p/p}^p$ and $\alpha_2 \beta_{p/p}^p$?

$$\begin{array}{ccccc}
H^0(q_2^{-1}Q/(q_0^\infty, q_1^\infty)) & \xrightarrow{\text{CSS}} & H^*(P,Q) & \xrightarrow[\cong]{\text{CESS}} & Ext_{\mathcal{A}}^2 \\
\text{Alg.}\downarrow\text{NSS} & & \text{Alg.}\downarrow\text{NSS} \quad \nearrow \Phi & & \downarrow\text{ASS} \\
H^0(v_2^{-1}BP_*/(p^\infty, v_1^\infty)) & \xrightarrow[\text{CSS}]{} & Ext_{BP_*BP}^2 & \xrightarrow[\text{ANSS}]{} & \pi_*(S^0)
\end{array}$$

$$\begin{array}{cccccc}
 & & 2q_1\xi_1, \quad b_1 & \qquad & 2q_1\xi_1 \cdot b_1^p & \xrightarrow[\cong]{\text{CESS}} \widetilde{\alpha}_2 b_1^p \neq 0 \\
 & & \text{Alg.}\downarrow\text{NSS} & & \text{Alg.}\downarrow\text{NSS} & \\
\frac{v_1^2}{p}, \quad \frac{v_2^p}{pv_1^p} & \xrightarrow[\text{CSS}]{} & \alpha_2, \quad \beta_{p/p} & & \alpha_2 \cdot \beta_{p/p}^p = 0 &
\end{array}$$

$d_{2p-1}(\beta_{p^2/p^2-1}) = \alpha_2 \beta_{p/p}^p$ and $\beta_{p^2/p^2-1}$ survives to $E_\infty$ imply $\alpha_2 \beta_{p/p}^p = 0$.

- $b_1^p = \beta_{p/p}^p \neq 0$ in $Ext_{BP_*BP}^{2p,*}(BP_*, BP_*)$, but $i_*(\beta_{p/p}) = 0$ in $Ext_{BP_*BP}^{2p,*}(BP_*, BP_*X)$

$$\cdots \longrightarrow Ext^{s-1}(BP_*\Sigma^q\overline{X}) \xrightarrow{\delta} Ext^s(BP_*) \xrightarrow{i_*} Ext^s(BP_*X) \longrightarrow \cdots$$

- We computed the $E_1$-term $E_1^{s,qp^3,u}$ of the SDSS subject to $s+u=2p$, which is generated by

$$\beta_1 h_{11}\gamma_2 b_{20}^{p-3} \qquad \beta_1\alpha_1 b_{20}^{p-3}\eta_p \qquad \beta^{\frac{p-1}{2}}\alpha_1\mathfrak{h}.$$

This gives a relation $\beta_{p/p} = \beta_1\mathfrak{g}$ and

$$\alpha_2\beta_{p/p}^p = \alpha_2\beta_1\mathfrak{g} = 0.$$

At prime $p=5$, $\beta_{5/5}^5 = \beta_1 x_{952}$ and $\alpha_2\beta_{5/5}^5 = 0$ (D. Ravenel's Green Book).

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# Conjecture

- Here we guess $\beta^p_{p/p} = \beta_1 h_{11}\gamma_2 b_{20}^{p-3}$ and

$$\begin{aligned}
&\beta^p_{p/p} = \beta_1 h_{11}\gamma_2 b_{20}^{p-3}\\
&\beta^p_{p^2/p^2} = \beta_1 h_{21} h_{11}\delta_3 b_{30}^{p-4}\\
&\ldots\\
&\beta^p_{p^i/p^i} = \beta_1 h_{i,1} h_{i-1,1}\cdots h_{11}\alpha^{(i+2)}_{i+1} b^{p-i-2}_{i+1.0}\\
&\ldots\\
&\beta_{p^{p-2}/p^{p-2}} = \beta_1 h_{p-2,1} h_{p-3,1}\cdots h_{11}\alpha^{(p)}_{p-1}
\end{aligned}$$

where $\alpha^{(i+2)}_{i+1}$ is the $i+2$-ed Greek letter elements.

# Conjecture

- For $i = 0, 1, \cdots, p-2$

$$\alpha_2\beta_{p^i/p^i} = \alpha_2\beta_1 h_{i,1} h_{i-1,1} \cdots h_{11} \alpha_{i+1}^{(i+2)} b_{i+1.0}^{p-i-2} = 0$$

and for $n = 1, 2, \cdots, p-1$, $\beta_{p^n/p^n-1}$ survives to $E_\infty$.

- There is the doomsday for $\beta_{p^n/p^n-1}$. If the doomsday for $V(n)$ is 50 years old, ($V(\frac{p+1}{2})$ does not exist), the doomsday for $h_0h_n$ is 100.

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# Conjecture

- For $i = 0, 1, \cdots, p-2$

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  and for $n = 1, 2, \cdots, p-1$, $\beta_{p^n/p^n-1}$ survives to $E_\infty$.

- There is the doomsday for $\beta_{p^n/p^n-1}$. If the doomsday for $V(n)$ is 50 years old, ($V(\frac{p+1}{2})$ does not exist), the doomsday for $h_0 h_n$ is 100.

**Conjecture**

For $n \geqslant p-1$, $\alpha_2\beta_{p^n/p^n} \neq 0$ and

$$d_{2p-1}(\beta_{p^{n+1}/p^{n+1}-1}) = \alpha_2\beta^p_{p^n/p^n}.$$

From $\beta_{p^p/p^p-1}$, $\beta_{p^n/p^n-1}$ does not exist and from $h_0h_{p+1}$, $h_0h_n$ does not exist.

*Thank you!*